Stability of Unstably Stratified Shear Flow in a Channel Under Non-Boussinesq Conditions

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Abstract

We investigate the linear stability of unstably stratified Poiseuille flow between two horizontal parallel plates under non-Boussinesq conditions. It is shown, that Squire’s transformation can be used to reduce the three-dimensional stability problem to an equivalent two-dimensional one. The eigenvalue problem, consisting of the generalized Orr-Sommerfeld equations, is solved numerically using an integral Chebyshev pseudo-spectral method. The influence of the non-Boussinesq effects on stability is studied. The dependence of the critical Rayleigh number on the Reynolds number and temperature difference parameter is obtained. As in the Boussinesq case, results show that the most unstable mode is that of longitudinal rolls. However, in contrast to the Boussinesq case, the rolls are highly distorted for large temperature differences. In addition, the critical Rayleigh number increases with the increase of the temperature difference and is independent of the Reynolds number.

1 Introduction

The deposition of thin inorganic films from precursors in the gas phase onto a solid substrate is a key element in a wide range of technological applications, including the fabrication of microelectronic circuits, and optical and magnetic recording media. Film thickness uniformity is generally critical in maintaining the same performance characteristics across each substrate and from sample to sample. Since it is well known that the efficiency of deposition is directly related to the temperature difference, an understanding of the nature of the non-Boussinesq mixed convection flow in a channel becomes essential to achieving uniform and efficient film growth. A detailed discussion of flow phenomena in chemical vapor deposition reactors as well a comprehensive list of references to theoretical and experimental works is given in an excellent review article by Jensen et al. [1].

If the flow and temperature fields between the two horizontal plates were fully developed, the phenomenon might be considered to be two-dimensional and the resulting deposition would be uniform. However, it is known that for some conditions the flow becomes unstable, resulting in steady or unsteady vortex rolls with axes aligned with the flow, transverse to the flow, or at some other angle in between. Thus the flow and temperature fields, and subsequently the uniformity of deposition, are completely affected by instabilities leading to such three-dimensional flows.

A typical channel reactor is usually a few centimeters high with an aspect ratio (width to height) of one to ten and a length of approximately one meter. Gas velocities are low, the Reynolds number is typically in the range $1 \sim 1600$, while the Rayleigh number varies between $50 \sim 10^7$ [1]. Consequently, a mixed convection flow results from the interaction between buoyancy and the imposed gas flow, and as a first approximation, the reactor can be modelled by the flow between two parallel infinite plates.

Expansion effects caused by density changes with heating of the gas play a major role in the flow behavior and can be modelled by the ideal gas law. Because of large temperature variations the Boussinesq assumption is not appropriate. Furthermore, it is necessary to include temperature variations in the transport and thermophysical properties. However since the Mach number is low ($Ma \lesssim 0.01$), the low Mach number approximation of Paolucci [2], which allows arbitrary density variation, can be used.

The effects of stratification on the stability of parallel shear flows, namely the interaction between the shear-driven mechanism of instability associated with shear flows and the buoyantly-driven instability associated with an adverse temperature gradient, have been extensively studied in the past. But most of the analytical/computational results to date are valid only for small temperature differences since the Boussinesq approximation is used (see for example [3]–[5]), or have limited applicability since only some of the properties are allowed to vary with temperature (see [6]). Gage and Reid [4] carried out the analysis for the Rayleigh–Bénard problem in the presence of a plane Poiseuille flow for $Pr = 1$, and more recently Fujimura and Kelly [5] investigated the linear stability of unstably stratified plane Poiseuille–Couette flow for $Pr = 0.51$. Thus the present paper can be considered as an extension of the above-mentioned works for large temperature differences.
2 Problem Definition

Consider the flow of a gas in a channel of height $H$ and unbounded in the horizontal direction. The bottom and top walls of the channel are maintained at temperatures $T_h^*$ and $T_c^*$, respectively, where $T_h^* \geq T_c^*$. The incoming flow can be characterized by a reference temperature $T_r$ and a pressure $p_r$. We non-dimensionalize the problem by reference quantities for length, velocity and temperature using the channel height $H$, the thermal diffusion speed $u_r = \alpha_r/H$, and the arithmetic average of the wall temperatures $T_r = (T_h^* + T_c^*)/2$, respectively. The problem is non-dimensionalized as follows (starred quantities are dimensional):

$$
\begin{align*}
  x_i^* &= Hx_i, \quad t^* = (H/u_r)t, \quad u_i^* = u_ru_i, \quad T^* = T_rT, \\
  P^* &= p_rp_r, \quad \rho^* = \rho_r\rho_r, \quad \Pi^* = \rho_ru_r^2\Pi, \quad (1) \\
  c_p^* &= c_p/c_{p_r}, \quad \mu^* = \mu_r\mu_r, \quad k^* = k_rk_r.
\end{align*}
$$

In the above definitions $\nu_r$ and $\alpha_r$ are the kinematic viscosity and thermal diffusivity respectively, $\beta_r$ is the coefficient of thermal expansion, $g$ is the magnitude of the gravitational field, $\gamma_r = c_p/c_{p_r}$ is the ratio of specific heats, and $\Delta T = T_h^* - T_c^*$. The subscript $r$ denotes that the properties are evaluated at the reference temperature and pressure. Note that for the non-Boussinesq regime our choice for the reference temperature will be shown to be crucial in avoiding a strong dependency of the critical Rayleigh number on the temperature difference. We will discuss this issue in detail at the end of section 5.

The problem evolves in time $t$ and can be described in terms of the velocity components $u_i = (u,v,w)$ in the $x_i = (x,y,z)$ directions, the density $\rho$, temperature $T$, and pressure $p$. The governing equations are statements of conservation of mass, momentum, and energy, with the addition of the ideal gas law. These equations, valid under low Mach number conditions, but allowing for arbitrary density variations, have been derived by Paolucci [2], and are given as follows:

$$
\begin{align*}
  \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} &= 0, \quad (2) \\
  \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_j}{\partial x_j} &= -\frac{\partial \Pi}{\partial x_i} + \frac{RaPr}{2\epsilon} \rho \rho_i + \rho T \frac{\partial \tau_{ij}}{\partial x_j}, \quad (3) \\
  \rho c_p \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) - \Gamma \frac{d \rho}{d t} &= \frac{\partial}{\partial x_j} \left( \frac{k}{\partial x_j} \frac{\partial \Pi}{\partial x_j} \right), \quad (4) \\
  P &= \rho T, \quad (5)
\end{align*}
$$

where $x$ is in the longitudinal direction, $y$ is in the vertical direction, $z$ is in the transverse direction, $\Pi = p^{(1)}/(\gamma_r Ma^2)$ is a reduced pressure which accounts for the hydrostatic and dynamic effects, $\rho^{(1)}$ is the second term in the Mach-number expansion of $p$ and is $O(Ma^2)$, $\epsilon = (0,-1,0)$ is the unit vector in the direction of gravity, and $\tau_{ij}$ is the viscous stress tensor given by

$$
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \mu \frac{\partial u_k}{\partial x_k}, \quad (6)
$$
where \( \delta_{ij} \) is the Kronecker delta. The independent dimensionless parameters appearing in the equations are

\[
Ra = \frac{\beta_r \gamma \Delta T H^3}{\nu_r \alpha_r}, \quad Pr = \frac{\nu_r}{\alpha_r}, \quad \epsilon = \frac{\Delta T}{2T_r}, \quad \Gamma = \frac{\gamma_r - 1}{\gamma_r}.
\]  

(7)

These parameters represent the Rayleigh and Prandtl numbers, and measures of overheating and fluid resilience. The governing equations are supplemented with equations describing transport properties.

The spatially uniform pressure \( P = p^{(0)}(t) \), appearing in the energy equation and the equation of state, represents the first term in the expansion of \( p \), and accounts for the change of the static pressure with time. The separation of the pressure components, holding under the small Mach-number approximation, is the essence of acoustic wave “filtering”, however this splitting introduces the extra unknown \( P(t) \). It can be shown, that since the channel is open, and because of the non-dimensionalization, \( P = 1 \), and assuming that the gas behaves as calorically perfect, we then have that

\[
\rho T = 1 \quad \text{and} \quad c_p = 1,
\]

and the solution is then independent of \( \Gamma \) (see equation (4)). To account for variations in transport properties, the dimensionless thermal conductivity and dynamic viscosity are given by the Sutherland-law forms

\[
k = T^{\eta/2} \left( \frac{1 + S_k}{T + S_k} \right), \quad \mu = T^{\eta/2} \left( \frac{1 + S_\mu}{T + S_\mu} \right),
\]

(9)

where White [7] gives the dimensional values of \( S_{k,\mu}^* = T_r S_{k,\mu} \) for a variety of gases along with their corresponding ranges of validity. Although the local Prandtl number is constant when \( S_k = S_\mu \), this simplification is not justified in many cases. The accuracy of these equations over a wide temperature range is discussed by Chenoweth and Paolucci [8].

Boundary conditions at the horizontal walls are given by

\[
u_i |_{y=0} = u_i |_{y=1} = 0, \quad T|_{y=0} = 1 + \epsilon, \quad T|_{y=1} = 1 - \epsilon.
\]

(10)

From the definition of \( \epsilon \) we note that \( 0 \leq \epsilon < 1 \), which corresponds to the temperature difference range of \( 0 < \Delta T < \infty \).

Due to an imposed constant pressure gradient in the longitudinal direction, the resulting flow has the average speed given by \( \bar{U} = < \rho \bar{u}^* > / < \rho > \), where \( < f > \) denotes the integration of \( f \) in the vertical direction between the plates. This independent velocity scale introduces the Reynolds number defined by

\[
Re = \frac{UH}{\nu_r}.
\]  

(11)

Note, that the Reynolds number does not appear explicitly in the equations, but it arises from an imposed longitudinal pressure gradient, and thus it gives a measure of the flow rate.
3 Basic Flow

3.1 Analytical Solution

If we assume the incoming flow to be laminar, a simple estimate shows that the flow becomes approximately fully developed at a distance $L_1 \sim HRe/10$ from the channel entrance. If the Rayleigh number is high enough, the flow becomes unstable and longitudinal rolls begin to develop (longitudinal rolls, as will be shown later, are the most unstable). The experimental results obtained by Chiu and Rosenberger [9] show that entrance length for the onset of buoyancy-driven convective instability can be estimated from the following relationship

$$L_2 = (0.65 \pm 0.05)H \left( \frac{Ra - Ra_c}{Ra_c} \right)^{-0.44 \pm 0.01} Re^{0.70 \pm 0.02} \text{ for } Ra > Ra_c,$$

where $Ra_c$ is a critical Rayleigh number for Rayleigh-Bénard convection. In cases of interest to us a large region $L_1 < L < L_2$, where the flow can be considered laminar and fully developed, exists. Then for such flow only the horizontal velocity component is non-zero $\bar{u}_i = (\bar{u}, 0, 0)$, and all non-zero temperature and velocity gradients are in the vertical direction. In addition, using the equation of state (8), the vertical component of the momentum equation (3) becomes

$$\frac{\partial \bar{\Pi}(x, y)}{\partial y} = -\frac{Ra Pr}{2\epsilon} \bar{\rho}(y).$$

Noting that the right-hand-side is a function of $y$ only, upon integration with respect to $y$ it becomes obvious that the pressure $\bar{\Pi}$ can be decomposed into the hydrodynamic and hydrostatic components

$$\bar{\Pi}(x, y) = \bar{\Pi}_d(x) + \bar{\Pi}_h(y).$$

Thus the steady-state energy and momentum equations reduce to

$$\frac{d\bar{\Pi}_d(x)}{dx} = Re Pr^2 \frac{d}{dy} \left( \bar{\rho} \frac{d\bar{u}}{dy} \right),$$

$$\frac{d\bar{\Pi}_h(y)}{dy} = -\frac{Ra Pr}{2\epsilon} \bar{\rho}(y),$$

$$\frac{d}{dy} \left( \bar{k} \frac{dT}{dy} \right) = 0,$$

$$\bar{\rho} \bar{T} = 1,$$

$$\bar{u} = \bar{u}_0 = \bar{u}_1 = 0, \quad \bar{T} = 1 + \epsilon, \quad \bar{T} = 1 - \epsilon,$$

where, using equations (9), $\bar{\rho} = \mu(\bar{T})$, $\bar{k} = k(\bar{T})$ and the basic flow solution is denoted by the bar. Note that in equation (14) we have made the Reynolds number explicit by using the average speed $U$ instead of the thermal diffusion speed $u_r$ to rescale the basic flow.

We observe from equation (14) that since the left-hand-side is only a function of $x$, while the right-hand-side is only a function of $y$, then both sides must be constant. Consequently, the following procedure is used in order to obtain the basic flow solution: first, equation (16)
is solved for $\bar{T}$, then the density is found using the equation of state (17), and subsequently, the hydrostatic pressure component is obtained from equation (15). Finally, the longitudinal velocity $\bar{u}(y)$ can be obtained by integrating (14). Constant property results can be obtained by setting $\bar{k} = \bar{\nu} = 1$, while results valid in the Boussinesq limit can be obtained by taking the limit $\epsilon \to 0$. Furthermore, the constant longitudinal pressure gradient can be easily related to the average velocity once $\bar{u}(y)$ is obtained. The solution is given explicitly in the appendix. In figure 1 we display the velocity and temperature profiles for several values of $\epsilon$, and for $Pr = 0.7$, $S_k = 0.5$, and $S_\mu = 0.357$ corresponding to nitrogen at $T_c = 300$ K as the carrier gas.

### 3.2 Numerical solution

Alternatively, the basic flow can be obtained numerically using an integral Chebyshev method. In this case, the problem (14)-(18) is solved over the domain $\hat{y} \in [-1, 1]$ ($\hat{y} = -1 + 2y$) for convenience. The essential ingredients of the integral Chebyshev method are given in [10].

The numerical integral operator $W_{ij}$, defined by

$$F(\hat{y}) = \int_{-1}^{1} f(\xi) d\xi \quad \iff \quad F(\hat{y}_i) = \sum_{j=1}^{N+1} W_{ij} f(\hat{y}_j)$$

with collocation points selected at $\hat{y}_i = \cos[(\pi - 1)/N]$ ($1 \leq i \leq N + 1$), is also given there.

Using the above definition and boundary conditions (18), equations (14) and (16) can be easily integrated resulting in

$$\bar{T}_i = (1 + \epsilon)\epsilon_i - 2\epsilon \sum_{j=1}^{N+1} \frac{W_{ij}}{\bar{\kappa}(\bar{T}_j)} \sum_{k=1}^{N+1} \frac{W_{jk}}{\bar{\rho}(\bar{T}_j)} e_j,$$

(19)

$$\bar{u}_i = \frac{1}{4Re Pr^2} \frac{d\Pi_d}{dx} \sum_{j=1}^{N+1} \frac{W_{ij}}{\bar{\mu}(\bar{T}_j)} \left\{ \sum_{k=1}^{N+1} \frac{W_{jk}}{\bar{\rho}(\bar{T}_j)} \frac{\sum_{l=1}^{N+1} W_{kl}}{\sum_{k=1}^{N+1} W_{kj}} e_j \right\},$$

(20)

where $\epsilon_i = 1$ for $i = 1, \ldots, N + 1$. The system of nonlinear equations (19) is solved using the IMSL routine NEQNF [11]. Subsequently, $\bar{u}_i$ is then explicitly evaluated using (20).

It should be noticed that in the Boussinesq case, since the variation of temperature is small, $\bar{k}(\bar{T}_i)$ and $\bar{\mu}(\bar{T}_i)$ reduce to constants equal to unity. Taking into consideration the fact that $\sum_{j=1}^{N+1} W_{ij} = (\epsilon_i + \hat{y}_i)/n!$, the numerical solution (19) and (20) reduce to the analytical linear temperature profile and parabolic Poiseuille flow:

$$\bar{T}_i = \epsilon_i - \epsilon \hat{y}_i,$$

(21)

$$\bar{u}_i = \frac{1}{8Re Pr^2} \frac{d\Pi_d}{dx} (\hat{y}_i^2 - \epsilon_i).$$

(22)

To ensure the accuracy of the proposed numerical method, the non-Boussinesq results were compared with the analytical solution obtained earlier. The comparison was done for the case $\epsilon = 0.3$, $S_k = 0.5$, $S_\mu = 0.357$. We have verified that the relative errors between the numerical solutions $\bar{T}_n$ and $\bar{u}_n$ given by (19) and (20) and the analytical solutions $\bar{T}_a$ and $\bar{u}_a$ given in the appendix decay exponentially with increasing number of collocation points, as expected. The errors become less than $10^{-12}$ when the number of collocation points reaches 22.
4 Perturbation Equations

Let \( \mathbf{u}_i = (\mathbf{u}, 0, 0), \mathbf{p}, \mathbf{P}, \mathbf{T} \) be the solution of the basic flow. Let us decompose the dependent variables into two parts: basic flow and disturbance. Then we can write

\[
\begin{align*}
    u_i &= \text{Re} Pr \bar{u}_i(y) + u'_i(x, y, z, t), \\
    \rho &= \bar{\rho}(y) + \rho'(x, y, z, t), \\
    \Pi &= \bar{\Pi}(x, y) + \Pi'(x, y, z, t), \\
    T &= \bar{T}(y) + T'(x, y, z, t).
\end{align*}
\]

(23)

Substituting (23) into equations (2)-(4), (6) and (8), subtracting the basic flow solution, and neglecting second order disturbance terms, we obtain the following set of equations for the disturbances:

\[
\frac{\partial u'_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( k \frac{\partial T'}{\partial x_j} + k' \frac{\partial T}{\partial x_j} \right),
\]

(24)

\[
\frac{\partial u'_i}{\partial t} + \text{Re} Pr \bar{u} \frac{\partial u'_i}{\partial x} + \text{Re} Pr \bar{\rho} \frac{\partial u'_i}{\partial y} = -\bar{T} \frac{\partial \Pi'}{\partial x_i} - \frac{\text{Ra} Pr}{2} \frac{T'}{T} \sigma_i + \text{Pr} \frac{\partial T'_{ij}}{\partial x_j},
\]

(25)

\[
\frac{\partial T'}{\partial t} + \text{Re} Pr \bar{u} \frac{\partial T'}{\partial x} + v' \frac{\partial T}{\partial y} = \bar{T} \frac{\partial}{\partial x_j} \left( k \frac{\partial T'}{\partial x_j} + k' \frac{\partial T}{\partial x_j} \right),
\]

(26)

where

\[
\tau'_{ij} = \bar{p} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \bar{p} \frac{\partial u'_k}{\partial x_k} + \text{Re} Pr \mu' \left( \frac{\partial \bar{\Pi}}{\partial x_i} + \frac{\partial \bar{\Pi}}{\partial x_i} \right)
\]

(27)

and we have used the fact that

\[
\mu' = \bar{\mu} T', \quad k' = \bar{k} T', \quad \rho' = -\frac{\bar{\rho}}{T} T' = -\frac{1}{T^2} T'
\]

where \( \bar{f}_T \equiv (df/dT) \). The boundary conditions for the perturbation equations (24)-(26) are:

\[
 u'_i|_{y=0} = u'_i|_{y=1} = T'|_{y=0} = T'|_{y=1} = 0.
\]

(28)

5 Stability Analysis

The basic flow is independent of \( t, x, \) and \( z \), so the Laplace transform with respect to \( t \) and the Fourier transform with respect to \( x \) and \( z \) may be taken to express a perturbation quantity

\[
f'(x, y, z, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{f}(y)e^{i\alpha x + i\gamma z + \sigma t} d\sigma d\alpha d\gamma,
\]

(29)

where \( f' \) is any disturbance quantity, \( \alpha \) and \( \gamma \) are longitudinal and transverse real wave numbers, \( \sigma = \sigma_R + i\sigma_I \) is the complex amplification rate, and the contour of the \( \sigma \)-integral is the Bromwich contour for the inversion of the Laplace transform. The real and imaginary parts of \( \sigma \) represent the amplification rate of the disturbance and the frequency, respectively.
The mode is stable, neutrally stable, or unstable depending on whether $\sigma_R$ is negative, zero, or positive. After Laplace and Fourier transforms, and making use of Squire's transformation

$$\ddot{\alpha} + \alpha \dot{\alpha} + \gamma \dot{\omega}, \quad \dot{\omega} = \dot{\bar{v}}, \quad \dot{\bar{v}} = \dot{w}, \quad \bar{T} = \bar{\hat{T}}, \quad \bar{\Pi} = \bar{\hat{\Pi}}, \quad \ddot{\alpha} = \alpha \dot{\alpha} + \gamma \dot{\omega},$$

$$\ddot{\alpha} = (\alpha^2 + \gamma^2)^1/2, \quad \gamma = \gamma, \quad \sigma = \sigma, \quad \bar{\epsilon} = \epsilon, \quad \bar{\rho}_T = \bar{\rho}_T, \quad \bar{\rho}_a = \bar{\rho}_a, \quad \bar{\rho}_e = \frac{\alpha}{\alpha} \bar{\rho}_e,$$

the system (24)-(26) reduces to

$$\ddot{T} (i\ddot{\alpha} \dot{\bar{v}} + D \ddot{\bar{v}}) = (i\ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \dot{\bar{v}} + \sigma) \ddot{T} + D \ddot{T} \ddot{\bar{v}},$$

$$\dot{\bar{v}} \ddot{T} \bar{\rho}_T \left( \bar{\rho}_T \left( \frac{1}{3} \ddot{\bar{v}} (i \ddot{\bar{v}} + 2 \ddot{\bar{v}} + \frac{4}{3} D \ddot{\bar{v}}) + i \ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \right) \right),$$

$$[\bar{T} \ddot{\bar{v}} \left( D^2 - \alpha^2 \right) - (i\ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \dot{\bar{v}} + \bar{\sigma})] \ddot{\bar{v}} = \bar{T} D \ddot{\bar{v}} - \bar{\rho}_a \bar{\rho}_T \ddot{\bar{v}} - \bar{\rho}_T D \ddot{\bar{v}} - \bar{\rho}_T \ddot{\bar{v}} \left( \frac{1}{3} \ddot{\bar{v}} (i \ddot{\bar{v}} + \frac{4}{3} D \ddot{\bar{v}}) + i \ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \right),$$

$$[\ddot{T} \ddot{\bar{v}} \left( D^2 - \alpha^2 \right) - (i\ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \dot{\bar{v}} + \bar{\sigma})] \ddot{T} = D \ddot{T} \ddot{\bar{v}} - \ddot{T} \left( \frac{1}{3} \ddot{\bar{v}} (i \ddot{\bar{v}} + \frac{4}{3} D \ddot{\bar{v}}) + i \ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \right),$$

$$[\ddot{T} \ddot{\bar{v}} \left( D^2 - \alpha^2 \right) - (i\ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \dot{\bar{v}} + \bar{\sigma})] \ddot{\bar{v}} = i \ddot{T} D \ddot{\bar{v}} - \ddot{T} \left( \frac{1}{3} \ddot{\bar{v}} (i \ddot{\bar{v}} + \frac{4}{3} D \ddot{\bar{v}}) + i \ddot{\alpha} \bar{\rho}_e \bar{\rho}_T \right),$$

where $D \equiv d/dy$. Equations (31)-(35), together with the boundary conditions

$$\ddot{\bar{v}} \big|_{y=0} = \ddot{\bar{v}} \big|_{y=1} = \ddot{T} \big|_{y=0} = \ddot{T} \big|_{y=1} = 0,$$

define the three-dimensional complex eigenvalue problem for the complex amplification rate $\bar{\sigma}$.

It can be seen that (31)-(34) are independent of $\ddot{\bar{v}}$, however (35) depends on all variables. For the purpose of simplifying the analysis it is necessary to formulate the following theorem.

**Theorem 1** Given the eigenvalue problem

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{\sigma} \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

with boundary conditions

$$\begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

where $L_{ij}$, $M_{ij}$, and $B_{ij}$ are linear operators, and $x$ and $y$ are vectors, then the problem (37)-(38) has two and only two families of solutions:
1. The first family is composed of eigenvalues and eigenvectors \( \hat{\sigma}^{(1)} \) and \( (x_1, y_1)^T \), where \( \hat{\sigma}^{(1)} \) and \( x_1 \) are obtained from the solution of the problem \( L_{11} x_1 = \hat{\sigma}^{(1)} M_{11} x_1 \) with boundary conditions \( B_{11} x_1 = 0 \) and \( y_1 \) is obtained from the solution of \( (L_{22} - \hat{\sigma}^{(1)} M_{22}) y_1 = -L_{21} x_1 \) with boundary conditions \( B_{22} y_1 = -B_{21} x_1 \).

2. The second family of eigenvectors is given by \( \hat{\sigma}^{(2)} \) and \( (0, y_2)^T \) obtained by solving the problem \( L_{22} y_2 = \hat{\sigma}^{(2)} M_{22} y_2 \) with boundary conditions \( B_{22} y_2 = 0 \).

**Proof.** Considering the structure of the eigenvalue problem (37)-(38), it follows that \( x \) does not depend on \( y \) and is the solution of the following eigenvalue problem \( L_{11} x = \hat{\sigma} M_{11} x \) with boundary conditions \( B_{11} x = 0 \). There are two and only two distinct solutions to this eigenvalue problem:

a) the nontrivial solution, corresponding to \( x_1 \) and \( \hat{\sigma}^{(1)} \); or

b) the trivial solution \( x_2 = 0 \).

The nontrivial solution \( x_1 \) gives the first family of eigenvalues and eigenvectors, while the trivial solution results in the second family.

Using the results of the above theorem, the eigenvalue problem (31)-(36) is exactly of the form as (37)-(38) with \( x = (\hat{u}, \hat{v}, \hat{\tilde{T}}, \hat{\Pi})^T \) and \( y = \hat{w} \). Now we wish to prove that for our case \( \hat{\sigma}^{(2)} R \leq 0 \) always, so that we only need to consider the eigenvalue problem associated with \( \hat{\sigma}^{(1)} \) to obtain the marginal stability surface. The eigenvalue problem associated with \( \hat{\sigma}^{(2)} \) is

\[
\left\{ \hat{\varpi} \hat{R} \hat{T} \hat{\Pi} \left( D^2 - \hat{\sigma}^2 \right) + \hat{\varpi} \hat{T} \hat{\Pi} D^2 \hat{T} D - \left( i \hat{\varpi} \hat{R} e \hat{\varpi} \hat{R} \hat{u} + \hat{\sigma}^{(2)} \right) \right\} \hat{w} = 0,
\]

or in divergence form

\[
\left\{ \left( D \hat{\varpi} D - \hat{\varpi} \hat{\sigma}^2 \right) - \frac{1}{\hat{\varpi} \hat{T}} \left( i \hat{\varpi} \hat{R} e \hat{\varpi} \hat{R} \hat{u} + \hat{\sigma}^{(2)} \right) \right\} \hat{w} = 0,
\]

Multiplying equation (40) by \( \hat{w}^* \) (the complex conjugate of \( \hat{w} \)), and integrating by parts over the height, we easily obtain

\[
\hat{\sigma}^{(2)} = -\hat{\varpi} \hat{R} \hat{T} \hat{\Pi} \hat{R} \hat{u} < \hat{\varpi} \hat{w}^2 / \hat{T} > \hat{T} > > 0,
\]

and

\[
\hat{\sigma}^{(2)} = -\hat{\varpi} \hat{R} \hat{T} \hat{\Pi} \hat{R} \hat{u} < \hat{\varpi} \hat{w}^2 / \hat{T} > \hat{T} > > 0.
\]

Thus the eigenvalue problem (31)-(34) with boundary conditions (36) constitutes a problem for the determination of the critical Rayleigh number. This eigenvalue problem is identical to the problem obtained without the use of Squire’s transformation with \( \gamma = \hat{w} = 0 \). Thus we can conclude that the three-dimensional problem is equivalent to a two-dimensional one at a smaller Reynolds number and the same Rayleigh and Prandtl numbers. Consequently, in order to obtain results for the three-dimensional problem we have to solve the equivalent two-dimensional one (31)-(34), and (36). From the solution of this problem we can then derive
by means of the transformation (30) the required solution of the three-dimensional problem. Note that throughout the literature (see for example [4], [5], [15]) it is common practice to use Squire’s transformation to obtain a reduced system of differential equations and then to examine only the eigenvalues of the reduced system to determine the stability of the basic flow. The additional eigenvalues associated with the semi-decoupled equation (analogous to (35)) are not considered. Hence the stability results for the reduced problem are valid if and only if the eigenvalues associated with the neglected equation have negative real parts, as demonstrated in our case. The use of the theorem is essential in a complete stability analysis.

The equivalent two-dimensional eigenvalue problem can be rewritten in matrix form by explicitly writing $\mathbf{L}_{11}$ and $\mathbf{M}_{11}$ as:

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
0 & A_{32} & A_{33} & 0 \\
A_{41} & A_{42} & A_{43} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\hat{\delta} \\
\hat{T} \\
\hat{\Pi}
\end{bmatrix}
= \hat{\sigma}
\begin{bmatrix}
B_{11} & 0 & 0 & 0 \\
0 & B_{22} & 0 & 0 \\
0 & 0 & B_{33} & 0 \\
0 & 0 & B_{43} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\hat{\delta} \\
\hat{T} \\
\hat{\Pi}
\end{bmatrix},
$$

(43)

where $A_{ij}$ are operators defined by

$$
\begin{align*}
A_{11} &= \hat{\mathcal{R}} \hat{T} \left[ \hat{\mathcal{L}} \left( D^2 - \frac{4}{3} \hat{\alpha}^2 \right) - i \hat{\alpha} \hat{\mathcal{R}} e^{\frac{\hat{\alpha}}{T}} + \hat{\mathcal{P}} D \hat{T} \right], \\
A_{12} &= -\hat{\mathcal{R}} e \hat{\mathcal{P}} D \hat{\mathcal{L}} + i \hat{\alpha} \hat{\mathcal{R}} \hat{T} \left( \frac{1}{3} \hat{\mathcal{P}} D + \hat{\mathcal{P}} T \hat{T} \right), \\
A_{13} &= \hat{\mathcal{R}} \hat{T} \left( \hat{\mathcal{P}} D^2 \hat{\mathcal{L}} + \hat{\mathcal{P}} D \hat{T} \right), \\
A_{14} &= -i \hat{\alpha} \hat{T}, \\
A_{21} &= \hat{\mathcal{R}} \hat{T} \left( \hat{\mathcal{L}} D - 2 \hat{\mathcal{P}} D \hat{T} \right), \\
A_{22} &= \hat{\mathcal{R}} \hat{\mathcal{Q}} \hat{T} \left[ \hat{\mathcal{L}} \left( D^2 - \hat{\alpha}^2 \right) - i \hat{\alpha} \hat{\mathcal{R}} e^{\frac{\hat{\alpha}}{T}} + \frac{4}{3} \hat{\mathcal{P}} D \hat{T} \right], \\
A_{23} &= \hat{\mathcal{R}} \hat{\alpha} \hat{T} \hat{\mathcal{P}} \frac{1}{2} \hat{T} + i \hat{\alpha} \hat{\mathcal{R}} \hat{\mathcal{P}} \hat{T} \hat{\mathcal{L}} \hat{\mathcal{L}}, \\
A_{24} &= -\hat{T} \hat{\mathcal{L}}, \\
A_{32} &= -\hat{\mathcal{P}} \hat{\mathcal{P}} \hat{T}, \\
A_{33} &= \hat{\mathcal{P}} \left[ \hat{\mathcal{L}} \left( D^2 - \hat{\alpha}^2 \right) - i \hat{\alpha} \hat{\mathcal{R}} \hat{\mathcal{P}} \hat{T} e^{\frac{\hat{\alpha}}{T}} + \hat{k} \left( 2 \hat{T} D \hat{T} + D^2 \hat{T} \right) + \hat{k} \hat{T} \left( \hat{T} D \right)^2 \right], \\
A_{41} &= i \hat{\alpha} \hat{T}, \\
A_{42} &= \hat{T} \hat{\mathcal{P}} \hat{T} - \hat{\mathcal{P}} \hat{T}, \\
A_{43} &= -i \hat{\alpha} \hat{\mathcal{P}} \hat{T} \hat{\mathcal{P}} \mathcal{L},
\end{align*}

(44)

and

$$
B_{11} = B_{22} = B_{33} = B_{43} = 1.
$$

(45)

Equation (43), together with boundary conditions (36) define the equivalent two-dimensional complex eigenvalue problem for the amplification rate $\hat{\sigma}$. 

10
As done in the numerical solution of the basic flow, we solve the eigenvalue problem in the domain $\bar{y} \in [-1, 1]$. Using numerical integral and differential operators given in [10] and [12], the eigenvalue problem (43) can be rewritten in discrete form as

$$AX = \sigma BX,$$

(46)

where $X = (x)_j$ is defined as

$$X = (\bar{u}_1', \ldots, \bar{u}_{N+1}', \bar{v}_1', \ldots, \bar{v}_{N+1}', \bar{T}_1', \ldots, \bar{T}_{N+1}', \bar{\Pi}_1, \ldots, \bar{\Pi}_{N+1})^T,$$

(47)

and $A = (L_{11})_{ij}$ and $B = (M_{11})_{ij}$ are the $[4(N + 1)] \otimes [4(N + 1)]$ matrices obtained from pseudo-spectral discretization of (44) and (45). The eigenvalues $\sigma$ are then obtained using the IMSL routine GVLCG [11].

In order to check the accuracy of the numerical solution, the stability results are compared with known values in the Boussinesq limit ($\epsilon \rightarrow 0$) for plane Poiseuille flow by taking $\hat{Ra} = 0$, for mixed convection flow where $\hat{Ra} \neq 0$ and $\hat{Re} \neq 0$, and for the Rayleigh-Bénard problem in both the Boussinesq and non-Boussinesq cases by taking $\hat{Re} = 0$ (see [13]–[19]). These comparisons are shown in tables 1-3. In table 1 it is noted that we require almost twice as many modes as Zebib [13] to obtain solutions to the same degree of accuracy for the pure Poiseuille flow. The reason for this is that while Zebib used a Chebyshev collocation method similar to ours, he also used a stream function formulation to obtain his results. Similarly, Brenier et al. [14] also used the stream function formulation in conjunction with the Tau-Chebyshev method to obtain their results. In the non-Boussinesq case the stream function can not be defined, thus the hydrodynamic pressure gradient is not easily eliminated from the problem. Retaining the pressure as an additional unknown results in the appearance of spurious eigenvalues related to the elliptic nature of the hydrodynamic pressure. Removal of the pressure from the formulation in the Boussinesq equations results in fewer spurious eigenvalues present in the problem, leading to greater accuracy of the solution with fewer modes. In table 2 the critical values of Rayleigh number and wave number for $Pr = 7$ and $Re = 100$ compare well with the results of Platten and Legros [15] for mixed convection. For the Rayleigh-Bénard problem we observe that the analytically obtained critical values of Rayleigh and wave numbers of $Ra_{c_0} = 1707.76178$ and $\alpha_{c_0} = 3.116324$ [16] are in excellent agreement with our values for $\epsilon \rightarrow 0$ and $\hat{Re} = 0$ given in table 3. Very limited information is available in the non-Boussinesq regime. For the weakly non-Boussinesq case, Busse [17] showed that the critical Rayleigh number should vary as

$$\frac{Ra_c - Ra_{c_0}}{Ra_{c_0}} = Ce^2 + \cdots,$$

where $C$ is a constant. Subsequently, in a numerical study of two-dimensional Rayleigh-Bénard convection in a closed cavity having aspect ratio of twenty, Paolucci and Chenoweth [18] obtained the value for the constant to be approximately 0.1832 for air. More recently, Fröhlich [19], performed a linear stability study for the non-Boussinesq Rayleigh-Bénard problem using the same property variations used by Paolucci and Chenoweth. He obtained the critical Rayleigh number for several values of $\epsilon$. His results are in excellent agreement with our results obtained for air for $\epsilon$ up to 0.6, and which can be approximated by

$$\frac{Ra_c - Ra_{c_0}}{Ra_{c_0}} = 0.1335e^2 + 0.2821e^4,$$

(11)
with a correlation coefficient of 0.99998.

Here, and in the results that follow, it will be noticed that the critical Rayleigh number depends quadratically on $\epsilon$ and thus does not change dramatically from the one obtained using the Boussinesq approximation. However, this insensitivity to $\epsilon$ is due to the judicious choice of the reference temperature corresponding to the mean of the wall temperatures. An even better choice would be the mass-averaged temperature of the basic flow and would have resulted in the same quadratic dependence on $\epsilon$ but with a slightly smaller leading coefficient. We did not make this choice because it is more complicated, and it would require that the basic flow solution be known before choosing the reference temperature. To emphasize this point we observe that had we chosen the reference temperature to correspond to the cold wall temperature as done in [20], then the range of $\epsilon$ would become $[0, +\infty)$, the value of $\epsilon = 0.6$ would become 3, and the corresponding critical Rayleigh number (based on this new reference temperature) would be approximately two orders of magnitude larger than that resulting from the Boussinesq approximation. Note that there are other possible choices for reference temperature such $T^*_H$ or $T^*$ ($y^* = H/2$) (see [21]). The first choice (analogous to the case where the cold wall temperature is taken as the reference temperature) would result in a high dependence of the critical Rayleigh number on the overheat parameter. The second choice leads to similar dependence on $\epsilon$ as obtained by our choice of the arithmetic average, but using a much more complicated normalization requiring the basic field solution.

Lastly, we note that in the above as well as all following discussions, in reporting our numerical stability results in the Boussinesq limit of $\epsilon \to 0$ we actually use the value of $\epsilon = 10^{-5}$. In addition, all the following numerical results are obtained using $N = 41$, even though for low values of Reynolds numbers the same accuracy could be obtained with less than half as many modes.

6 Results and Discussion

As it was stated earlier, transformation (30) can be used to reduce the three-dimensional stability problem to an equivalent two-dimensional one. The precise relationship between the two- and three-dimensional problems follows immediately from the fact that Rayleigh and Prandtl numbers remain invariant under the transformation, Since $\alpha/\tilde{\alpha} = \cos \lambda$, where $\lambda$ is the angle between the transverse direction and the horizontal wave number, then the relationship between the two and three-dimensional Reynolds numbers can be written as $\tilde{Re} = Re \cos \lambda$. Once the stability boundary for the equivalent two-dimensional problem is obtained, the family of stability boundaries for the three-dimensional problem can be easily generated by varying the angle $\lambda$. The marginal stability curves for the equivalent two-dimensional problem are presented in figure 2 for $\tilde{\epsilon} \to 0$ (the Boussinesq case), $\tilde{\epsilon} = 0.3$, and $\tilde{\epsilon} = 0.6$. The flow is unstable above each of the curves for the appropriate value of $\tilde{\epsilon}$. The results displayed in the figure are presented quantitatively in table 3.

We note that for two-dimensional disturbances the critical Rayleigh number increases with the increase of $\tilde{\epsilon}$. Another interesting observation can be made is that for large Reynolds numbers the critical Rayleigh number behaves approximately as the Reynolds number to the $4/3$ power. The three-dimensional stability results for different $\epsilon$ are presented in figures 3. The curves labeled $\lambda = 0$ and $\lambda = 90$ correspond to the marginal stability curves for transverse
and longitudinal rolls, respectively. It is now clear that the flow is unstable for $Ra > Ra_c$, where the critical Rayleigh number $Ra_c = Ra_{|\lambda = 0}$ is independent of the Reynolds number. Since for any stable rolls with $Ra > Ra_c$ and with $\lambda_1 < 0$ there are rolls inclined at $\lambda_2 > \lambda_1$ that are unstable, it then follows that longitudinal rolls correspond to the most unstable mode. The nature of their instability is purely thermal in origin since, as it was pointed out, the critical Rayleigh number is independent of the Reynolds number, and it can be easily shown that the equations prior to using Squire’s transformation reduce exactly to the eigenvalue problem for the non-Boussinesq Rayleigh-Bénard problem when $\lambda = 90$.

The dependence of critical Rayleigh number $Ra_c$ on the overheat parameter $\epsilon$ is presented in figure 4 and quantitatively in table 4. We can see that the critical Rayleigh number increases monotonically with increase of the overheat parameter. Furthermore, with a correlation coefficient of 0.99991, the relative critical Rayleigh number for $\epsilon$ up to 0.6 can be approximated by

$$\frac{Ra_c - Ra_{c0}}{Ra_{c0}} = 0.0321\epsilon^2 + 0.2291\epsilon^4.$$  \hspace{1cm} (48)

As can be seen in figure 4, this equation provides for an excellent approximation of the data over the complete range of $\epsilon$. Note that in general the coefficients depend of $Pr$, $S_k$, and $S_\mu$ which in turn vary depending on the gas and on the reference temperature. The values given above correspond to our specific choices of these parameters.

It is noticed that the critical longitudinal wave speed of the most unstable mode for the equivalent two-dimensional problem $\hat{c}_c$ is equal to the three-dimensional one $c_c$, since it is independent of the roll angle $\lambda$:

$$\hat{c}_c = -\frac{\sigma_{lc}}{\alpha Re Pr} = -\frac{\sigma_{lc}}{\alpha Re Pr} = c_c.$$  \hspace{1cm} (49)

In figure 5 it can be easily seen that the wave speed is close to that of the mean flow when the Reynolds number is low enough, but it increases rapidly and approaches the maximum speed of the basic flow at higher Reynolds numbers. For future reference, we denote the interval of Reynolds numbers where the rapid changes occur as the transition region. Note that the dimensional critical wave speed approaches zero when the Reynolds number goes to zero, since $\hat{c}_c$ (and thus also $c_c$) is normalized by the average speed of the basic flow. This result is true for all $\hat{c}$ and is consistent with the known Rayleigh-Bénard result for $\hat{c} \to 0$.

From figure 6 we can see that in the transition region the critical wavenumber dependence changes from being approximately constant for lower Reynolds numbers to approximately logarithmic growth for large Reynolds numbers. The rapid changes in the transition region are due to the fact that with increase of Reynolds number the most unstable perturbation mode becomes more localized in the neighborhood of maximum velocity of the basic flow. By looking at the kinetic energy balance of that mode, it can be shown that the Reynolds stress $-\frac{1}{2} Re Pr < \overline{\rho \hat{u} \hat{v}} >$ always has a stabilizing influence, but it is the least stabilizing in the vicinity of the velocity maximum. This can be seen more clearly by looking at the marginal stability curves (a), (d), and (e) given in figure 7 and the corresponding eigenfunctions shown in figure 10.

From the marginal stability curves shown in figures 7–9 we also see the appearance of at least two local minima as the Reynolds number is increased. Note that multi-peaked neutral curves were also observed by Fujimura and Kelly [5] in the Boussinesq limit. Their
presence is due to shear-buoyancy interaction. The nature of each local minimum can be clearly understood by looking at figures 10 and 11, displaying the instantaneous velocity and temperature disturbances for points labelled a-j in figures 7-9. Note that the abscissas are normalized by the appropriate wavelength. In figures 10 (a-c) we can see the formation of longitudinal rolls for $\hat{Re} = 0$. For $\hat{c} \to 0$ the flow has symmetry with respect to the mid-plane, while with increase of temperature difference the center of the rolls shift towards the cold wall and the symmetry is lost.

Figures 10 (a,d,e) illustrate the effect of increasing Reynolds number for $\hat{c} \to 0$. With the increase of Reynolds number the rolls are more sheared in the longitudinal direction and the formation of heart-shape patterns is observed. These rolls become more confined near the cavity mid-height as the Reynolds number increases. In the case of large temperature differences, the rolls are similarly shaped, but loose the symmetry and with the increase of $\hat{c}$ move towards the cold wall along with the basic flow velocity maximum. By looking at the kinetic energy balance of the most unstable mode it can be shown that the Reynolds stress has an increasingly stabilizing influence with increase of Reynolds number.

The eigenfunctions shown in figures 10 (e) and 11 clarify the physical nature of the multiple minima observed in the marginal stability curves displayed in figures 7-9. It can be shown that the marginal stability curve for $\hat{Re} = 0$ is the limit for the curves corresponding to $\hat{Re} > 0$ as $\hat{c} \to 0$. In addition, it can be noticed from figure 7 that, for a fixed $\hat{Re} \gtrsim 200$, the lowest (in wavenumber) local minimum occurs close to the marginal stability curve corresponding to $\hat{Re} = 0$. Subsequently, the eigenfunction corresponding to such minimum is expected to be unaffected by the Reynolds stress. This fact can be further confirmed by looking at the kinetic energy balance of the disturbance, and is further evidenced by the similarity of eigenfunctions shown in figures 11 (f) and (g) corresponding to different Reynolds numbers. Figures 11 (h-j) are the eigenfunctions corresponding to the intermediate minima labeled (h-j) on the marginal curves shown in figure 7. They illustrate the effect of the shear-driven mechanism in the overall stability picture. As the Reynolds number increases along the dashed line shown in figure 7, the disturbances become more localized at the walls. Although not shown here, the kinetic energy balance indicates that for $\hat{Re} \lesssim 400$ the Reynolds stress contribution is stabilizing, while for $\hat{Re} \gtrsim 400$ it becomes destabilizing. The Reynolds stress contribution to the kinetic energy balance increases monotonically and it dominates the buoyancy contribution for $\hat{Re} \gtrsim 2200$. This middle branch is ultimately responsible for the shear-driven instability associated with pure Poiseuille flow.

The increase of $\hat{c}$ strengthens the buoyantly-driven instability and weakens the shear-driven one. Even though the quantitative behavior and the extent of the interactions change, the qualitative competition between buoyant and shear driven mechanisms remain the same as discussed above. The fundamental reason for the quantitative change is due to the loss of symmetry arising from non-Boussinesq effects. As noted earlier, the largest velocity and temperature gradients of the basic flow occur near the cold wall. Subsequently, the eigenfunctions of the disturbance field also shift towards the cold wall. For large values of $\hat{c}$, this shift can be so dramatic for large wavenumbers that the whole disturbance is localized in the neighborhood of the cold wall. For $\hat{c} = 0.6$, this behavior is illustrated by the eigenfunction shown in figure 12 corresponding to point (k) on the marginal stability diagram in figure 9.

We conclude by summarizing that for the range of Reynolds numbers studied, the forced component of the flow always has a stabilizing influence on the two-dimensional buoyant
instability. We recall that the flow is most unstable to three-dimensional disturbances. As in the Boussinesq case, the results show that the most unstable mode is that of longitudinal rolls. However, in contrast to the Boussinesq case, the rolls are highly distorted for large temperature differences. In addition, the critical Rayleigh number increases with the increase of the temperature difference and is independent of the Reynolds number. We point out that the two-dimensional disturbances are not mainly of academic interest since, as shown by Platten and Legros [15] in the Boussinesq limit, these disturbances are preferred in narrow channels for a finite range of Reynolds number. In addition, we also note that due to the asymmetry of the basic flow, it is expected that non-Boussinesq effects will give rise to a subcritical bifurcation. The nonlinear analysis necessary for its study will be the subject of future work.

Acknowledgment

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A Analytical Solution for the Basic Flow

The solution of equation (16) with boundary conditions (18) is given as follows in terms of

\[ \tau = \left( \frac{T}{S_k} \right)^{1/2}. \]  \hfill (50)

The temperature is related to \( y \) via

\[ y = \frac{f(\tau_h) - f(\tau)}{f(\tau_h) - f(\tau_c)}, \]

and the vertical heat flux is given by

\[ \bar{q} = \beta [f(\tau_h) - f(\tau_c)], \]

where

\[ \beta = 2(1 + S_k)S_k^{3/2}, \quad \text{and} \quad f(\tau) = \frac{1}{3} \tau^3 - \tau + \tan^{-1}(\tau). \]

Integrating equation (14) and using equations (9), (18), and (50), we obtain

\[ \bar{u}(\tau) = u_c \left\{ [Y(\tau_h) - Y(\tau)] - [Y(\tau_h) - Y(\tau_c)] \frac{F(\tau_h) - F(\tau)}{F(\tau_h) - F(\tau_c)} \right\}, \]

where

\[ u_c = -\frac{1}{Re Pr^2} \frac{d\Pi_d}{dx} \frac{S_k \beta \left( 1 + S_k \right)}{\bar{q}^2 \left( 1 + S_k \right)}. \]
\[ Y(\tau) \equiv Z(\tau) + (\hat{S} - 1) [L(\tau) + I(\tau)], \]

\[ F(\tau) \equiv \tau^2 + (\hat{S} - 1) \ln(1 + \tau^2), \]

\[ \hat{S} = \frac{S_k}{S_k}, \]

\[ Z(\tau) \equiv \frac{2}{15} \tau^5 - \frac{2}{3} \tau^3 - \tau + (1 + \tau^2) \tan^{-1} \tau, \]

\[ L(\tau) \equiv \frac{2}{9} \tau^3 - \frac{8}{3} \tau + \frac{8}{3} \tan^{-1} \tau, \]

\[ I(\tau) \equiv \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k+1}(2^{2k} - 1)}{(2k+1) \cdot (2k)!} B_{2k} \cdot (\tan^{-1} \tau)^{2k+1}. \]

The terms \( B_{2k} \) are Bernoulli numbers which can be generated for \( k = 1, 2, 3, \ldots \) from

\[ B_{2k} = \frac{1}{(2k+1)2^{2k}} \left[ 2k - \sum_{n=1}^{k-1} 2^{2n} \binom{2k+1}{2n} B_{2n} \right], \]

where

\[ \binom{r}{m} = \frac{r!}{m!(r-m)!} \]

are the binomial coefficients.

References


Authors' address: O. V. Vasilyev and S. Paolucci, Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556-5637, U.S.A.
Table 1: The most unstable eigenvalue $\hat{\sigma}/3\hat{Re}\hat{Pr}$ of plane Poiseuille flow ($\hat{Ra} = 0$, $\hat{\varepsilon} \to 0$) for $\hat{\alpha} = 2$, $\hat{Pr} = 1$, $\hat{Re} = 13333$.

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Table 2: Critical values for mixed convection flow using $N = 41$.

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Table 3: Critical values of wave number $\hat{\alpha}_c$, Rayleigh number $\hat{Ra}_c$, and longitudinal wave speed $\hat{c}_c$ for $\hat{Pr} = 0.7$ and $\hat{c} \to 0$, $\hat{c} = 0.3$, and $\hat{c} = 0.6$.

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<td>$4.16845 \times 10^3$</td>
<td>1.129</td>
</tr>
<tr>
<td>70</td>
<td>3.439</td>
<td>$6.87380 \times 10^3$</td>
<td>1.124</td>
</tr>
<tr>
<td>100</td>
<td>3.226</td>
<td>$1.36838 \times 10^4$</td>
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<td>$3.80882 \times 10^4$</td>
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<tr>
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<td>7.750</td>
<td>$6.54628 \times 10^4$</td>
<td>1.384</td>
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<tr>
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<td>9.207</td>
<td>$1.29512 \times 10^5$</td>
<td>1.417</td>
</tr>
<tr>
<td>700</td>
<td>10.304</td>
<td>$2.02877 \times 10^5$</td>
<td>1.434</td>
</tr>
<tr>
<td>1000</td>
<td>11.606</td>
<td>$3.26436 \times 10^5$</td>
<td>1.448</td>
</tr>
<tr>
<td>1500</td>
<td>13.287</td>
<td>$5.60523 \times 10^5$</td>
<td>1.460</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$10^{-5}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ra_c$</td>
<td>1707.76</td>
<td>1708.46</td>
<td>1710.90</td>
<td>1716.22</td>
<td>1726.54</td>
<td>1745.42</td>
<td>1778.69</td>
</tr>
</tbody>
</table>
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