The Role of Nonlinear Source Terms in Geophysics

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Abstract. Correct interpretation of results becomes an important issue when dealing with nonlinear phenomena of Solid Earth geophysics. Direct linearization of nonlinear processes can cause sufficient mistakes and even cardinally change the interpretation. In this paper a simple and effective procedure for structural recognition and visualization of nonlinear potential sources is presented. This procedure is based on parametric analysis of corresponding transients: switch and localized waves. Conditions for the formation of transitional waves and properties of these waves have been studied. The proposed approach is illustrated on a simple example of potential linear thermal source with nonlinear cubic disturbance. The procedure presented in this paper can be applied for detection of viscous heating and nonlinear heat diffusion in complicated media.

Introduction

There is a growing recognition of the importance of nonlinear phenomena in many branches of geophysics. It has recently been shown that such phenomena as earthquakes, volcanoes, plate tectonics, core motions and their reflection in geophysical fields, as well as propagation of geophysical fields in nonuniform geological media are the nonlinear processes (e.g. [Herouani and Kelly, 1991; Hofmeister, 1999; Newman et al., 1994; Beck and Schlögl, 1995; Rowlands, 1995; Keitis-Borok, 1990; Starin et al., 2000; Vasilyev et al., 1997]). At the same time it is still a common practice to solve complex geophysical problems by removing visible nonlinear effects and reducing the problem to a linear one. Such a linearization can sufficiently eliminate useful effects and even cardinally change the notion of targets. Therefore, many important geophysical problems should be solved using special nonlinear procedures.

It is widely recognized that a study of thermal behavior of the Earth is one of the most speculative branches of geophysics. This is caused, mainly, by the absence of a reliable information about heat sources and mechanisms of heat transfer and mixing. Nevertheless, thermal field detection is widely used to search useful minerals (first of all, oil and gas), to investigate near surface and deep geological structure, and to perform seismological prognosis [Newman et al., 1994; Eppelbaum et al., 1996; Main, 1996; Sharma, 1997].

The problem of influence of strongly nonlinear source on transitional dynamics is practically not investigated in geophysics. At the same time the problem can have important value in detection of viscous heating and nonlinear heat diffusion. In the present paper we discuss some peculiarities of the problem appearing in thermal field investigations.

Problem Definition and Discussion

A process of thermal wave propagation can be described by the following evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u + f(u),$$  \hspace{1cm} (1)

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator, $f(u)$ is a nonlinear function of a thermal source. Let us assume the following form of the thermal source:

$$f(u) = \alpha u + \epsilon g(u),$$  \hspace{1cm} (2)

where $\alpha$ and $\epsilon$ are some parameters, $g(u)$ is the nonlinear function which describes disturbances of the studied thermal field. Function $g(u)$ can be presented both as conventional deterministic function and random distribution function.

If $\epsilon = 0$, we obtain a linear equation corresponding to the conventional problem with a linear source term:

$$\frac{\partial u}{\partial t} = \Delta u + \alpha u.$$  \hspace{1cm} (3)

A case of $\epsilon \neq 0$ correspods to the situation when the normal thermal field is disturbed by anomalous sources. A nonlinear disturbance under specific conditions can cause an appearance of a shock wave even for extremely small values of the parameter $\epsilon$. This effect can be used for recognition of the desired target. On the other hand,
Eq. (1) under special conditions admits an appearance of transient solutions: localized or switch (kink) waves. Note that properties of both localized and shock waves are determined by characteristics of the source function \( f(u) \). Consequently, this fact allows us to estimate some parameters of the studied medium.

It was noted earlier that nonlinear processes often can be approximated by cubic nonlinearity [Friedman, 1991; Grimshaw et al., 1997; Kosevich and Kovalev, 1989; Wold, 1974]. In this paper we limit our consideration to the nonlinearity of the type \( \phi(u) = u^3 \), since it allows us to obtain analytical solutions. Other types of nonlinearity can be considered as well. The temperature source term of the form \( f(u) = \alpha u + cu^3 \) allows solutions of the traveling wave type:

\[
\psi(z, y, z, t) = \psi(hx + ky + k_z z + vt),
\]

(1)

where \( v \) is the wave propagation speed and \( \vec{k} = (k_x, k_y, k_z) \) is the unit vector in the direction of wave propagation. Self-similar solutions of such type were considered in detail in Barenblatt [1996].

Substitution of Eq. (4) into Eq. (1) yields:

\[
\psi'(s) = \psi''(s) + \alpha \psi(s) + \epsilon \psi^3(s),
\]

(5)

where \( \alpha = k_x x + k_y y + k_z z + vt \) and we used the fact that \( k_x^2 + k_y^2 + k_z^2 = 1 \). Next we will introduce non-conventional definitions for transition waves. In these definitions values \( \epsilon_1 \) and \( \epsilon_2 \) can be either finite or infinite (\( -\infty, +\infty \)) as in the traditional definitions [Eppelbaum and Kardashev, 1998].

**Definition 1** Differentiable solution \( \psi(s) \) of the Eq. (5) is defined as switch (kink) wave if the solution satisfies the following conditions:

a) \( \lim_{h \to \infty} \psi(h) = \psi_1 \) and \( \lim_{h \to -\infty} \psi(h) = \psi_2 \), where \( \psi_1, \psi_2 \) are constant solutions of the Eq. (1). These solutions are the nontrivial roots of the algebraic equation \( \alpha u + \epsilon u^3 = 0 \) given by

\[
\psi_1 = -\sqrt{-\frac{\alpha}{\epsilon}}, \quad \psi_2 = \sqrt{-\frac{\alpha}{\epsilon}}.
\]

(6)

**Definition 2** Differentiable solutions \( \psi_i(s) \) \( (i = 1, 2) \) of the Eq. (5) are defined as localized waves if these solutions satisfy the following conditions:

a) \( \lim_{s \to \infty} \psi_i(s) = \lim_{s \to -\infty} \psi_i(s) = 0 \);

b) \( \lim_{s \to \infty} \frac{d}{ds} \psi_i(s) = \lim_{s \to -\infty} \frac{d}{ds} \psi_i(s) = 0 \);

c) \( \psi_i(0) = \psi_i^0 \) \( (i = 1, 2) \).

The values \( \psi_i^0 \) \( (i = 1, 2) \), determining the amplitude of the localized waves, are roots of the potential function \( F(u) = \frac{1}{2} \alpha u^2 + \frac{1}{4} cu^4 \), i.e.,

\[
\psi_i^0 = \sqrt{-\frac{2\alpha}{\epsilon}}, \quad \psi_i^0 = \sqrt{-\frac{2\alpha}{\epsilon}}.
\]

(7)

The solution \( \psi_1(s) \) is defined as localized wave of positive polarity and \( \psi_2(s) \) as localized wave of negative polarity. Without loss of generality let us consider the localized wave of positive polarity. We suppose that the potential function \( F(u) \) satisfies to condition \( \alpha \epsilon < 0 \). Functions presented in Fig. 1(a) \( (\epsilon > 0, \alpha < 0) \) and Fig. 1(b) \( (\epsilon < 0, \alpha > 0) \) correspond to the localized and switch waves, respectively. The principal structures of the localized and switch waves are shown respectively in Figs. 2(a) and 2(b).

It was shown in Kardashev [1999] that transitive waves (both switch and localized waves) corresponding to the source function of type (2) are the steady-state solutions \( \psi(s) \) of Eq. (5). Thus, the transitional solutions satisfy the following ordinary differential equation:

\[
\psi'' + \alpha \psi + \epsilon \psi^3 = 0.
\]

(8)

This is a Duffing equation and its localized waves (elliptical Jacobi functions) can be written using exact expressions. The solutions for both switch and localized waves are respectively given by

\[
\psi_i(s) = \sqrt{-\frac{\alpha}{\epsilon}} \tanh \left( \sqrt{\frac{\alpha}{\epsilon}} s \right),
\]

(9)

\[
\psi_i^0(s) = \pm \sqrt{-\frac{2\alpha}{\epsilon}} \sinh \left( \sqrt{-\alpha} s \right), \quad i = 1, 2,
\]

(10)

where \( \tanh x = e^x - e^{-x} \) and \( \sinh x = \frac{e^x - e^{-x}}{2} \). For details of derivation we refer to [Nayfeh, 1973; Kosevich and Kovalev, 1989]. Note that the main parameters (amplitude and width) of both switch and localized waves can be identically determined by the structural parameters of the source term. In addition the amplitudes of the transitive solutions increase with the decrease of \( \epsilon \), which means that the corresponding solutions of Eq. (1) are shock waves when \( \epsilon \rightarrow 0 \). Visualization of these effects can be used for the nonlinear source recognition.

Equation (1) can be generalized to include the effect of nonlinear diffusion. Such models are widely used for description of thermal propagation processes in media with nonlinear coefficients of thermal conductivity [Beck and Schlögl, 1995; Hofmeister, 1999; Natalie and Sulusti, 1996; Starin et al., 2000; Wheatcraft and Guberman, 1991]. We will consider the following physically justified generalization of equation (1) with different types of thermal conductivity \( k_i \)

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[ k_i \left( \frac{\partial u}{\partial x_i} \right) \frac{\partial u}{\partial x_i} \right] + f(u),
\]

(11)

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[ k_2(u) \frac{\partial u}{\partial x_i} \right] + f(u).
\]

(12)

Figure 1: Potential function \( F(u) \) of the source \( f(u) = \alpha u + cu^3 \): (a) \( \alpha < 0, \epsilon > 0 \), (b) \( \alpha > 0, \epsilon < 0 \).
Equation (11) was used to describe strongly nonlinear heat catalytic processes in absorbing-reacting media [Diaz, 1985], while equation of type (12) can be used to describe the heat wave propagation in porous media [Kamin and Vazquez, 1991].

Equation (11) is a strongly nonlinear thermal evolution equation of a gradient type and can be rewritten as

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} G_i \left( \frac{\partial u}{\partial x_i} \right) + f(u).$$

(13)

This equation can be investigated by variational methods (e.g., [Diaz, 1985; Kardashev, 1999]). On the other hand, the transformation

$$\frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right) = \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i^2}$$

(14)

implies that equation (12) includes nonlinear terms discussed in the recent publications [Hofmeister, 1999; Starin et al., 2000]. It should be noted that Barenblatt [1996] used the intermediate asymptotics for entering similar nonlinearities.

It was shown in Kardashev [1999] that if

$$\tilde{k}_i(u) = O\left(|u|^\alpha_i\right); \quad \tilde{k}_i(u) = O\left(|u|^\beta_i\right); \quad f(u) = O\left(|u|^\gamma\right) \quad u \to 0$$

(15)

and $\gamma > 1, \alpha_i > \gamma - 1, \beta_i > \gamma - 1$ then the transients have finite-localized structure. In particular, for the sources of the form $f(u) = au + cu^3$, $(\alpha \neq 0)$, transients have the finite-localized or periodic structure if the following conditions are satisfied

$$\alpha_i > 0; \quad \beta_i > 0.$$  

(16)

In this case the oscillating fronts may appear and localized waves of compaction type [Rosenau and Hyman, 1993] may arise. In other words, localized waves have a compact support and switch waves are either periodic or have finite support as well. Examples of compaction, periodic switch waves, and switch wave with compact support are given in Figs. 3, 4(a) and 4(b) respectively. Note the main difference between Figs. 2(a) and 3 and Figs. 2(b) and 4(b) is the compact support of compactons and finite switch waves. The compactons and localized/periodic switch waves present more visible images and allow us to estimate not only parameters of the source but also the intensity of the nonlinear diffusion. $\tilde{k}_i(u) = Du^{\beta_i}$ ($\beta_i > 0$).

Conditions (15) or (16) guarantee that the values $s_1$ and $s_2$ in Definitions 1 and 2 are finite numbers. The width $\lambda$ of a transient wave can be easily calculated from the values of $s_1$ and $s_2$, i.e., $\lambda = s_2 - s_1$. The value of $\lambda$ is determined by other parameters of the problem and can be viewed as some function

$$\lambda = \lambda \left(\alpha_i, \beta_i, A, B, \sqrt{-\alpha \over \epsilon}\right).$$

(17)

Values $\alpha_i, \beta_i, A, B, \sqrt{-\alpha \over \epsilon}$ can be considered as structural parameters, which could be determined from measurements. These parameters describe structural properties of the strongly nonlinear sources and nonlinear diffusion.

The property of finite localization of the transients shows the possibility of wave front propagation in the finite time. It presents the physically important phenomena allowing to recognize and control the time and spatial bounds of the arised patterns.

Another possible application of the suggested approach can be illustrated on a sufficiently simple example. Thermal method of geophysical prospecting is used for solving many problems. One of such problems is detection and recognition of anomalies from oil pools. Generation and migration of oil are complicated nonlinear processes [Hunt, 1990]. If the pool occurs at comparatively small depth, the heat front potential generated by such a source can be approximated by a function of type $f(u) = au + cu^3$, $(\alpha > 0, \epsilon < 0)$ (at least at the interval limited by two maximal points – see Fig. 1(b)). Taking into account that in this case a strong thermal gradient can be observed, Eq. (12) describing the heat-wave propagation can be used. Assuming that $\alpha_i > 0$, we can construct exact periodic switch wave solution [Kardashev, 1999]. Visualization of such parameters as wave amplitude and wavelength, will allow us to reestablish by the above-mentioned method the desired structural parameters of the investigated source.

It should be noted that recently [Bouzat and Wio, 1998] obtained an exact expression for the nonequilibrium potential of Lyapunov functional type for a three-component reaction-diffusion system. Contrary to the approach of Bouzat and Wio [1998] we consider a problem of the direct estimation of the deterministic parameters of a thermal source.
Conclusions

In this work we propose a novel approach for recognition and visualization of nonequilibrium potential sources with the use of the corresponding transitional wave analysis. Main goal of this investigation is to show that transitional waves deterministically characterize the structural parameters of the nonlinear sources. Results of this investigation allow us to conclude that

1. Parameters of the considered transitional waves characterize the structure and main characteristics of the geophysical (thermal) source function. A set of the observed data of transient time values or other parameters of the transients give a possibility to estimate and even to reestablish the desired characteristics.

2. The models with nonlinear diffusion and sources can be effectively applied to the solution of various geophysical problems (first of all, for geothermal investigations). The transitional waves obtained using these models will allow investigators to determine and estimate not only parameters of the studied sources but also the nonlinear diffusion.

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References


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