Local spectrum of commutation error in large eddy simulations

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In this Brief Communication we present a new mathematical tool, which we call local spectrum analysis, that can be used to obtain information about local spectral content of the commutation error in large eddy simulations and its dependence on the filter shape and the non-uniformity of the filter width. To illustrate these theoretical findings, the local commutation spectrum analysis is applied to the results of 256\textsuperscript{3} direct numerical simulation of forced homogeneous turbulence at Re\textsubscript{\textit{x}} = 168. The results confirm strong dependence of the spectral content of the commutation error on the filter shape: the spectrum is wide for smooth filters like Gaussian, while for filters, that are close to the sharp cut-off, the spectral content is localized. It is also demonstrated that the amplitude of the commutation error is linearly proportional to the filter width stretching factor. © 2004 American Institute of Physics. [DOI: 10.1063/1.1637605]

In large eddy simulation (LES) of turbulent flows the dynamics of the large scale structures are computed, while the effect of the small scale turbulence is modeled. The differential equations describing the space–time evolution of the large scale structures are formally derived by applying a low-pass filter with non-uniform filter width to the Navier–Stokes equations. For an incompressible flow the filtered Navier–Stokes equations, written in terms of filtered quantities, take the following form:

\[
\frac{\partial \bar{u}_i}{\partial x_i} = \left[ \frac{\partial u_i}{\partial x_i} \right],
\]

\[
\frac{\partial \bar{u}_i}{\partial t} + \left[ \frac{\partial \bar{u}_i}{\partial x_j} \right] = - \frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\text{Re}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \left[ \frac{\partial u_i u_j}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\text{Re}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} \right],
\]

where the square bracket denotes the commutation operator given by

\[
\left[ \frac{\partial F}{\partial x_j} \right] = \frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i}.
\]

The filtered convective term \( \bar{u}_i \bar{u}_j \) is unknown in LES and is typically decomposed into the convective term \( \bar{u}_i \bar{u}_j \) that can be computed and the remainder, called sub-grid scale (SGS) stress, which should be modeled

\[
\bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j - \frac{\bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j}{\tau_{ij}}.
\]

In order to derive LES equations from Eqs. (1), (2), and (4) it is commonly assumed that the differentiation and filtering operations commute. With this assumption we obtain the classical LES equations

\[
\frac{\partial \bar{u}_i}{\partial t} = 0,
\]

\[
\frac{\partial \bar{u}_i}{\partial t} + \left[ \frac{\partial \bar{u}_i}{\partial x_j} \right] = - \frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\text{Re}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j},
\]

Commutation is generally satisfied if the filter has a constant width. However, this assumption is invalid if the filter width is not uniform—as in the case of wall-bounded flows—unless special filter operators are constructed. Recently a new class of commutative filters for both structured\textsuperscript{a} and unstructured\textsuperscript{b, c} grids has been developed. With these filters the differentiation and filtering operations commute to an \textit{a priori} specified order of filter width. The desired commutation error is achieved by constructing filters in such a way that the filter moments \( M^{(n)}(x) = \int x^n \mathbb{f}(y) \, dx dy \) satisfy the following constraints:

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that the commutation error is negligible compared to the sub-
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local spectrum
a novel mathematical tool that can be used to analyze the
spectrum will be global and will not contain any local infor-
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an above can be viewed as a practical tool for constructing dis-

Leading order commutation error analysis described
above can be viewed as a practical tool for constructing dis-
crete filters that commute with finite difference operators to an
a priori specified order of filter width. However, the lead-
ing order error analysis by itself is not sufficient to guarantee
that the commutation error is negligible compared to the sub-
grid scale stress, since it does not use any information about
spectral content of the analyzed signal.4,5 Due to the presence
of significant energy in the high frequency portion of the
LES spectrum, the commutation error could be considerable
and in some cases even comparable with the subgrid scale
stresses. In addition, the application of the standard or win-
dowed Fourier transform for the analysis of the commutation
error will result in a spectrum that does not differentiate be-
the different effects that influence it, namely the filter
shape, filter width, and its spatial variation. Furthermore, the
spectrum will be global and will not contain any local informa-
spatial variation in the spectral content of the com-
mutation error.

The objective of this Brief Communication is to present
a novel mathematical tool that can be used to analyze the
local spectral content of the commutation error (hereafter
referred as the local spectrum of the commutation error) and
its dependence on the filter shape and the non-uniformity of
the filter width.

We begin by introducing the local spectrum analysis in
one-dimensional space and then extend it to three spatial
dimensions. Let us consider a one-dimensional filter of con-
stant shape but variable width. The continuous filtering op-
can be written as
where, in contrast to Eq. (8), the filter shape is fixed through-
out the domain, while the filter width changes as a function of
position. Note that the filter can be either symmetric or
asymmetric. Substituting the Fourier integral
into Eq. (10), changing the order of integration, and integra-
the resulting equation with respect to \( y \) we obtain

\[
\bar{\phi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}(k)e^{ikx}dk, \tag{12}
\]

where \( \bar{\phi}(k) \) is the Fourier transform of the filter function
given by

\[
\bar{G}(k) = \int_{-\infty}^{\infty} G(\xi)e^{-ik\xi}d\xi. \tag{13}
\]

Now comparing Eqs. (11) and (12) we can see that the struc-
ture of the equations is the same, except that the term in the
curly brackets in Eq. (12) has implicit spatial dependence.
Thus, the local Fourier transform of the filtered quantity \( \hat{\phi} \)
can be defined as

\[
\hat{\phi}(k; x) = \hat{G}(\Delta(x)k)\hat{\phi}(k). \tag{14}
\]

Note that in order to calculate \( \hat{\phi} \) at a specific loca-
tion, \( x_0 \), we only need to know the local spectrum \( \hat{\phi}(k; x_0) \).
Now the meaning of the local Fourier transform is clear: it refers
to the location in space, where the transform is taken, and gives
the information about the spectral content of the commuta-
tion error. Also it reflects the fact that the filter width is a
function of location. Performing analogous analysis for the
commutation error it can be shown that

\[
\frac{d\phi}{dx}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{\Delta(x)} \frac{d(\Delta(x))}{dx} K(\Delta k)\hat{\phi}(k) \right\} e^{ikx}dk, \tag{15}
\]

where \( k_\Delta = \Delta(x)k \) and the transfer function \( K(\Delta k) \) is
defined by

\[
K(\Delta k) = -k \left. \frac{d\hat{G}(k)}{dk} \right|_{k=0}. \tag{16}
\]

Analogously to Eq. (14), the local one-dimensional spectrum
of the commutation error is defined as

\[
\frac{d\phi}{dx}(x) = \frac{1}{\Delta(x)} \frac{d(\Delta(x))}{dx} \hat{K}(\Delta k)\hat{\phi}(k). \tag{17}
\]

Now the effect of filtering is clearly seen: filter width stretch-
factor, \( [1/\Delta(x)][d(\Delta(x))/dx] \), affects the amplitude of
the commutation error, while the filter shape and width affect
the local spectral content. The effect of the filter shape on the
spectrum of commutation error is demonstrated in Fig. 1,
where transfer functions \( \hat{G}(k_\Delta) \) and \( \hat{K}(k_\Delta) \) are shown for
a variety of filters. The wavenumber \( k_\Delta \), which effectively
defines the filter width, is marked by a dashed vertical line. It
is important to note that the closer the filter to the sharp
cut-off, the more localized in wavenumber space the spectral
content of the commutation error. Also note that the commuta-
tion error is exactly zero, when the width of the filter is
constant throughout the domain.

The one-dimensional analysis can be easily extended to
three spatial dimensions. In particular, it can be shown that
for three-dimensional filters of constant shape and variable
width the commutation error is given by
Another important point that we want to elaborate on, is a physical interpretation of the local spectrum concept. We use the word “local” to emphasize the fact that the local spectrum analysis provides the information about the spectral content of the commutation error, which is associated with a particular point in space, where the filter shape, width, and filter width stretching factor are known. In general, when this analysis is applied to an inhomogeneous case, the windowed Fourier energy spectrum changes from location to location. However, if local spectrum analysis is applied to the case, where the global spectrum exists and the filter width varies only in one spatial direction, then the only two parameters that affect the spectral content of the commutation error are the filter width and filter width stretching factor. In this case for any two points that have the same filter width, the local spectral content of the commutation error will differ only in the amplitude that is determined by the filter width stretching factor. Thus in this case, the location for the local spectrum analysis does not need to be explicitly specified.

To demonstrate the effect of commutation error in LES, we apply the local commutation error analysis to the results of 256³ direct numerical simulation (DNS) of forced homogeneous turbulence at Re₉ = 168. It should be emphasized that the homogeneous turbulence field is used for illustration purposes only and, as was mentioned earlier, the analysis can be applied in a more general setting using the windowed Fourier transform. Figure 2 shows the local one-dimensional spectrum of commutation error for the Gaussian and the discrete M₇ filter (see Fig. 1). These two filters are chosen because the Gaussian filter is smooth, while the M₇ filter is close to the sharp cut-off filter. The wavenumber k₁, which effectively defines the filter width in the x₁ direction, is

\[
\frac{\partial \phi}{\partial x_i}(x) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \sum_{l=1}^{3} \frac{\partial (\Delta_l(x))}{\partial x_i} \right] \phi(k) \hat{k}_i \nonumber
\]

where \( \Delta_l(x) \) is the filter width in the \( x_i \) direction, \( k = (k_1, k_2, k_3)^T \) and \( k_3 = (\Delta_1(x)k_1, \Delta_2(x)k_2, \Delta_3(x)k_3)^T \) are three-dimensional wave vectors, \( \hat{\phi}(k) \) is the Fourier transform of \( \phi(x) \), \( \hat{G}(k) \) is the Fourier transform of the filter function, and the transfer function \( \hat{K}(k) \) is defined by

\[
\hat{K}(k) = -k \frac{\partial \hat{G}(k)}{\partial k_1}.
\]
marked by a dashed vertical line. Note that the filter width, \( \Delta_1 \), that was used in these calculations can be easily extracted from \( k^*_x \) and \( k^*_1 \). Also note that at any given point in space the filter width and the stretching factors are sufficient for the calculation of the local spectrum of commutation error [see Eq. (18)]. Consequently, the actual filter width distribution is not required for the calculation of the local spectrum of commutation error. Note that the local spectrum of the commutation error shown in Fig. 2 assumes a unity filter width stretching factor. In contrast, the local windowed Fourier spectrum for the SGS force, \( \partial \tau_{11}/\partial x_1 \), at any given point requires knowledge of the actual filter width distribution. However, the global spectrum of the exact SGS force is a good approximation for the slowly varying filter width case. For that reason the global one-dimensional spectrum of the exact SGS force is also shown in Fig. 2 for reference. The exact SGS force is obtained by filtering the DNS data using Gaussian and \( M \) filters (see Fig. 1) with the same filter width that is used in the calculation of the local spectrum of the commutation error. The results confirm strong dependence of the spectrum of commutation error on the filter shape: the spectrum is global for smooth filters like Gaussian, while for filters, that are close to the sharp cut-off, the spectrum is localized. In addition, the local spectrum analysis confirms that in order for the commutation error to be negligible compared to the SGS force, the filter width stretching factor should be considerably below unity.

In conclusion we want to emphasize that the local spectrum analysis, proposed in this Brief Communication, provides new theoretical insights on the spectral content of the commutation error. It gives a LES practitioner the guidelines on how to decrease the effect of the commutation error and provides a new way to look at the spectral content of the commutation error.

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